The Killing Form

- a bilinear form B is invariant if $\forall x,y,z \in \mathfrak{g}$, $B(\mathrm{ad}_x y, z) + B(y, \mathrm{ad}_x z) = 0$
- Lemma: B invariant & $I \subset \mathfrak{g}$ ideal then $I^\perp = \{y \in \mathfrak{g} : B(I,y) = 0\}$
  - note: $I^\perp \cap I \neq 0$ in general
- $B_R(x,y) = \mathrm{Tr}_R(\rho(x)\rho(y))$ for a representation R
  - B_R is a symmetric, invariant bilinear form on $\mathfrak{g}$
  - for $\mathfrak{gl}(n,\mathbb{C})$: $\mathrm{tr}([x,y]z) + \mathrm{tr}(y[x,z]) = \mathrm{tr}(xyz - yxz + yxz - yzx) = 0$

- Thrm: if for some R, B_R is non-degenerate then, $\mathfrak{g}$ is reductive
- Thrm: $\mathfrak{gl}(n,k)$, $\mathfrak{sl}(n,k)$, $\mathfrak{so}(n,k)$, $\mathfrak{u}(n)$, $\mathfrak{su}(n)$, $\mathfrak{sp}(n,k)$ are reductive
  - $\mathfrak{sl}(n,k)$, $\mathfrak{so}(n>2,k)$, $\mathfrak{su}(n)$, $\mathfrak{sp}(n,k)$ are semisimple
  - $\mathfrak{gl}(n,k) = k \cdot \mathrm{id} \oplus \mathfrak{sl}(n,k)$ & $\mathfrak{u}(n) = i\mathbb{R} \cdot \mathrm{id} \oplus \mathfrak{su}(n)$
  - for $\mathfrak{gl}(n)$: $B_\square(x,y) = x^{\mu\nu} y_{\nu\mu}$ non-degenerate

    decomposition follows from orthogonality in $B_\square$:

    $\mathrm{Tr}(\mathrm{id} \cdot x) = \mathrm{Tr}\, x = 0$ for $x \in \mathfrak{sl}(n,k)$

- Killing form $K(x,y) = B_{ad}(x,y) = \mathrm{tr}(\mathrm{ad}_x, \mathrm{ad}_y)$
  - explicitly for $\mathfrak{sl}(2,\mathbb{C})$:

    basis of $e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $h = e^T$, $f = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$

    $\Rightarrow [e,f] = h$, $[h,e] = 2e$, $[h,f] = -2f$

    so, $\mathrm{ad}_e = \begin{pmatrix} 0 & -2 & \\ & 0 & 1 \\ & & 0 \end{pmatrix}$, $\mathrm{ad}_h = \begin{pmatrix} 2 & & \\ & 0 & \\ & & -2 \end{pmatrix}$, $\mathrm{ad} f = \begin{pmatrix} 0 & & \\ -1 & 0 & \\ & 2 & 0 \end{pmatrix}$

    $\Rightarrow K = \begin{pmatrix} 0 & & 4 \\ & 8 & \\ 4 & & 0 \end{pmatrix}$ in the $\mathrm{ad}_e, \mathrm{ad}_h, \mathrm{ad}_f$ basis

    so, $K = B_{ad} = 4B_\square$
- Cartan's criterion of: a) solvability, $\mathfrak{g}$ solvable iff $K([\mathfrak{g},\mathfrak{g}], \mathfrak{g}) = 0$

  b) semisimplicity, $\mathfrak{g}$ semisimple iff K non-degenerate

  Prf: $\Rightarrow$ a) by Lie's thrm, $\exists$ a basis s.t. ad_x is upper triangular

  so, $\mathrm{tr}(\mathrm{ad}_x \mathrm{ad}_y) = 0$

  $\Leftarrow$ b) $\mathfrak{g}$ reductive & $\forall x \in \mathfrak{z}(\mathfrak{g})$, $\mathrm{ad}_x = 0$ so, $x \in \ker K$

  $\therefore \mathfrak{z}(\mathfrak{g}) = 0$ & $\mathfrak{g}$ is semisimple

  $\Rightarrow$ b) $I = \ker K$ an ideal of $\mathfrak{g}$ (Lemma)

  $K|_I$ is the Killing form of I which therefore vanishes

  $\therefore I$ is solvable

  $\mathfrak{g}$ semisimple $\Rightarrow I = 0$ so K is non-degenerate

  $\Rightarrow$ a) Relies on Jordan decomposition:

  $A: V \to V$ linear, $A = A_s + A_n$ with $[A_s, A_n] = 0$
  - A_s semisimple: if $AW \subset W$ then $\exists W'$ s.t. $V = W \oplus W'$ & $AW' \subset W'$
  - A_n nilpotent: $A^n = 0$ for some n

The Cartan Decomposition

- $x \in \mathfrak{g}$ is semisimple or nilpotent if ad_x is as an operator on $\mathfrak{g}$
- Thrm: if $\mathfrak{g}$ semisimple then any $x \in \mathfrak{g}$, $x = x_s + x_n$ where:
  - x_s is semisimple & x_n is nilpotent
  - $[x_s, x_n] = 0$ & $[x,y] = 0$ then $[x_s, y] = 0$
  - this is a general form of the Jordan decomposition
- Thrm: any semisimple $\mathbb{C}$ Lie algebra has non-trivial semisimple elements
  - otherwise all nilpotent + Engel's thrm

- $\mathfrak{h} \subset \mathfrak{g}$ is a toral subalgebra if it is commutative & has only semisimple elements
- ⚠ note: from now on $\mathfrak{g}$ is a complex, semisimple Lie algebra
- Thrm: $\mathfrak{h} \subset \mathfrak{g}$ toral & B a non-degenerate, invariant, symmetric, bilinear form
  1) $\mathfrak{g} = \bigoplus_{\alpha \in \mathfrak{h}^*} \mathfrak{g}_\alpha$ for $\mathfrak{g}_\alpha$ eigenspace of $\mathrm{ad}_h, h \in \mathfrak{h}$, $\mathrm{ad}_h x = \langle \alpha, h \rangle x$ $\forall x \in \mathfrak{g}_\alpha$

     in particular $\mathfrak{h} \subset \mathfrak{g}_0$
  2) $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta}$
  3) $\alpha + \beta \neq 0$, $\mathfrak{g}_\alpha, \mathfrak{g}_\beta$ orthogonal w.r.t. B
  4) $\forall \alpha$, B defines non-degenerate $\mathfrak{g}_\alpha \otimes \mathfrak{g}_{-\alpha} \to \mathbb{C}$

  Prf: 1) by definition, ad_h diagonalizable for $h \in \mathfrak{h}$

  simultaneously so since toral algebras commute

  2) let $\mathfrak{g} = \oplus \mathfrak{g}_\lambda$ s.t. $(\mathrm{ad}_x - \lambda \cdot \mathrm{id})^n |_{\mathfrak{g}_\lambda} = 0$ for $n > \dim \mathfrak{g}_\lambda$

  consider $[\mathfrak{g}_\lambda, \mathfrak{g}_\tau]$, $y \in \mathfrak{g}_\lambda$, $z \in \mathfrak{g}_\tau$

  $(\mathrm{ad}_x - \lambda - \tau)[y,z] = [(\mathrm{ad}_x - \lambda)y, z] + [y, (\mathrm{ad} - \tau)z]$

  so, $(\mathrm{ad}_x - \lambda - \tau)^n [y,z] = \sum_k \binom{n}{k} [(\mathrm{ad}_x - \lambda)^k y, (\mathrm{ad}_x - \tau)^{n-k} z]$

  $= 0$ for $n > \dim \mathfrak{g}_\lambda + \dim \mathfrak{g}_\tau$

  3) $B([h,y],z) + B(y,[h,z]) = 0 = (\langle h,\alpha \rangle + \langle h,\beta \rangle) B(x,y)$

  so, $B(x,y) \neq 0 \Rightarrow \beta = -\alpha$
- Lemma: 1) B restricted to $\mathfrak{g}_0$ is non-degenerate

  2) $x \in \mathfrak{g}_0$ then $x_s, x_n \in \mathfrak{g}_0$

  3) $\mathfrak{g}_0$ is a reductive subalgebra
- Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ is toral and equal to its centralizer $C(\mathfrak{h}) = \mathfrak{h}$
  - $\mathfrak{sl}(n,\mathbb{C})$: $\mathfrak{h} = \{\text{traceless diagonal matrices}\}$ is commutative & semisimple $\therefore$ toral

    fact from lin alg: $[\mathfrak{h}, x] = 0 \Rightarrow x \in \mathfrak{h}$

    so, $\mathfrak{h}$ is a Cartan subalgebra
- Thrm: maximal toral subalgebras are Cartan

  Prf: $\mathfrak{g} = \oplus \mathfrak{g}_\alpha$ eigenspaces of some maximal toral $\mathfrak{h} \subset \mathfrak{g}$

  $\forall x \in \mathfrak{g}_0$, $\mathrm{ad}_x|_{\mathfrak{g}_0}$ is nilpotent:

  otherwise, $\mathrm{ad}_{x_s}|_{\mathfrak{g}_s} \neq 0$ so, $x_s \in \mathfrak{h}$

  since $x_s \in \mathfrak{g}_0$, $[\mathfrak{h}, x_s] = 0$ so $\mathfrak{h} \oplus \mathbb{C} \cdot x_s$ would be toral

  contradicts maximality of $\mathfrak{h}$

  $\left.\begin{array}{l} \text{Engel's thrm} \Rightarrow \mathfrak{g}_0 \text{ is nilpotent} \\ \text{Lemma} \Rightarrow \mathfrak{g}_0 \text{ is reductive} \end{array}\right\} \mathfrak{g}_0 \text{ is commutative}$

  ad_{x_n} nilpotent $\Rightarrow \mathrm{ad}_{x_n} \mathrm{ad}_y$ nilpotent $\forall y \in \mathfrak{g}_0$

  so, $\mathrm{tr}_{\mathfrak{g}}(\mathrm{ad}_{x_n} \mathrm{ad}_y) = 0$

  then, non-degeneracy of Killing form $\Rightarrow x_n = 0$

  so, $\forall x \in \mathfrak{g}_0$, x is semisimple $\therefore C(\mathfrak{h}) = \mathfrak{g}_0 \supset \mathfrak{h}$ is toral

  thus $\mathfrak{h} = C(\mathfrak{h})$ is Cartan
- Cor: all complex semi-simple $\mathfrak{g}$ contain a Cartan subalgebra
- Thrm: for a complex, semisimple $\mathfrak{g}$,

  $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in R} \mathfrak{g}_\alpha$ where $\mathfrak{h} \subset \mathfrak{g}$ is a Cartan subalgebra

  with "root subspaces" $\mathfrak{g}_\alpha = \{x : \mathrm{ad}_h x = \langle \alpha, h \rangle x \ \forall h \in \mathfrak{h}\}$

  and "root system" $R = \{\alpha \in \mathfrak{h}^* \setminus \{0\} : \mathfrak{g}_\alpha \neq 0\}$